Coordination and Social Distancing: Inertia in the Aggregate Response to COVID-19

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March 27, 2020

Abstract

Social distancing is critical to slow the spread of COVID-19. However, social distancing in the wake of COVID-19 has been frustratingly slow and inadequate. Here we use a game theoretic model to show that, when a new and rare virus, like COVID-19, emerges, the aggregate level of social distancing has inherent inertia, and that clear national public statements are essential in reducing that inertia and adjusting the public's behavior to the new, optimal level of social distancing. Novel infectious diseases abruptly change the appropriate level of social distancing, leaving individuals uncertain about how to act. Inertia arises in such a setting because individuals care about conforming to social norms (e.g., it is awkward to refuse a social invitation or work request) and the previous level of social distancing provides a focal point to coordinate behavior. Clear and consistent national statements about the new optimal level of social distancing enable individuals and communities to coordinate on new norms of behavior, reducing inertia and moving the society closer to the optimum. Such national statements generate a beneficial over-reaction from the public that offsets the over-weighting of past experience. National communications are preferable to communications through local governments or employers when the optimal levels of social distancing are highly correlated over-time and when individuals are poorly-informed about changes in the optimal level of social distancing. Our results show the utility of game theoretic models in disease control and public health policy.

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Motivated by the COVID-19 outbreak, we study a model of social distancing in which people care about two things: (1) engaging in the correct amount of social distancing and (2) conforming to the behavior of other people (i.e., adhering to social norms). When citizens care about these two considerations, and are uncertain about each others' beliefs about the right amount of social distancing, aggregate social distancing exhibits significant inertia. In particular, following a large positive shock to the right amount of social distancing (e.g., because of the outbreak of a contagious virus), aggregate social distancing will be far below the social optimum, even if individuals' information accurately reflects this shock on average.

Can the government improve this situation? The cause of inertia is that people over-weight their common knowledge about the past level of social distancing because doing so allows them to coordinate on a similar level of social distancing (i.e., follow social norms)—e.g., it is awkward for both parties when one refuses to shake the other's hand. Public statements from prominent leaders (e.g., presidential speeches) about the new optimal level of social distancing reduce the inertia by allowing people to coordinate their actions around new, more appropriate norms.

Given the over-weighting of public messages, we ask, contrary to standard intuitions, whether leaders should communicate more privately—e.g., via local governments or employers—rather than publicly. The over-weighting of highly public, national messages is beneficial when it off-sets counterproductive inertia. National communication is therefore peferable to local communication when optimal social distancing are highly correlated over-time and when individuals are poorly-informed about changes in optimal social distancing. Our results demonstrate how leaders can improve social distancing outcomes through a communications strategy that balances competing

We also ask whether such public statements are the best way to provide information that moves people toward the right level of social distancing. Given the over-weighting of public messages, would it be better for leaders to communicate more privately—e.g., via local governments or employers—rather than publicly. The over-weighting of highly public, national messages is beneficial when it off-sets counterproductive inertia. National communication is therefore peferable to local communication when optimal social distancing are highly correlated over-time and when individuals are poorly-informed about changes in optimal social distancing.

What sort of decisions might our model represent in the context of the COVID-19 crisis? The key features of the model are that: (1) people are uncertain of the right level of social distancing and are uncertain of what others believe, (2) the right level of social distancing exhibits serial correlation, and (3) there is a desire for conformity—the less other people are social distancing, the less I want to socially distance in order to conform to norms.

One important kind of application concerns social gatherings. The advent of COVID-19 made it undesirable, from the perspective of social welfare, for people to participate in events such as St. Patrick's Day celebrations, spring break trips and parties, and the like. But people were uncertain how serious a threat COVID-19 really was. And, to the extent that they believed others might think the threat relatively minimal and therefore continue to gather, they too had incentives to behave this way. The result was undesirable inertia—people continued past practices in ways that were harmful to the social welfare.

Similar arguments hold for social practices such as hand shaking, kissing, and other forms of physical social greetings. It is awkward to refuse to shake an offered hand—as was evident in a recent presidential press briefing—creating a force for social conformity of the sort we model. The uncertainty about others' views on COVID-19 created unfortunate inertia in physical social greetings.

The model also applies, at least in some professional settings, to the choice to continue going to the office in the shadow of the COVID-19 threat. If employees believe that coming in signals commitment or ambition, this creates social pressure for employees to continue coming to work if their managers or supervisors are. In the face of such pressure, the model suggests, there will be inertia that keeps people coming into the office at inefficiently high levels, especially during the early days of a viral outbreak.[1]

1 Model

We build on the canonical framework and results discussed in Angeletos and Lian (2016).[2] There is a continuum of citizens indexed by $i \in [0,1]$, interacting over time, indexed by $t = 0, 1, \ldots$. In each period t, each citizen must take an action $a_{it} \in \mathbb{R}$, which captures the degree of social distancing. A higher action corresponds to a higher level of social distancing. Absent concerns for conformity, the right action for each citizen is θ_t. But, in each period, each citizen cares about targeting this right action *and* about conforming to the average action that others take in that period, $A_t = \int a_{it}\, di$. This generates a complementarity: if a citizen believes that others will do little social distancing, this raises that citizen's incentive also to do less social distancing. This captures, among other things, social pressure and the cost of deviating from the norms of behavior in the society. In particular, a citizen's payoff in period t is:

$$-(1-\alpha)(a_{it} - \theta_t)^2 - \alpha(a_{it} - A_t)^2, \tag{1}$$

where $\alpha \in (0,1)$ is the citizens' relative weight on conformity.

The right action, θ_t, follows a random walk: $\theta_t = \theta_{t-1} + u_t$, where $u_t \sim iidN(0, \sigma_u)$. Citizens do not observe θ_t, but each citizen observes a signal of the right action: $x_{it} = \theta_t + \epsilon_{it}$, where $\epsilon_{it} \sim iidN(0, \sigma_\epsilon)$. Throughout, we assume that the noise and fundamentals are independent from each other in the standard manner.

[1] In this setting, there could also be congestion externalities that we leave unmodeled—as fewer people start going into the office, the risk of being infected in the office goes down.

[2] This literature builds on the seminal work of Morris and Shin (2002).

Citizen i observes x_{it} in period t, and θ_{t-1} becomes public in period t.[3] Citizens discount future payoffs by δ, and each citizen maximizes the expected sum of discounted period payoffs.

2 Analysis

We will think of the normative goal as each individual choosing the right level of aggregate social distancing, $a_{it} = \theta_t$, so that the aggregate level of social distancing is also right, $A_t = \int a_{it}di = \theta_t$. This would be the objective of a policy maker who aggregates individual payoffs, but puts no weight on social conformity (i.e., sets $\alpha = 0$).[4] In light of this, we say that any reduction in the expected quadratic distance between individual actions and the right action, $E[\int(a_{it} - \theta_t)^2 di]$, is a *normative improvement.*

Our first result concerns the optimal level of social distancing under this normative criterion. Because the right action is uncertain, the normatively optimal individual actions—i.e., the actions that minimize $E[\int(a_{it} - \theta_t)^2 di]$—involve citizens choosing their best estimate of the right action, $a_{it} = E[\theta_t|x_{it}, \theta_{t-1}] = \beta x_{it} + (1 - \beta)\theta_{t-1}$, where $\beta = \frac{\sigma_u^2}{\sigma_u^2+\sigma_\epsilon^2}$.

Proposition 1 ***(Normative Benchmark)*** *If citizens did not care about conformity ($\alpha =$ 0), the aggregate action would be: $A_t = \theta_t + \beta u_t$, where $\beta = \frac{\sigma_u^2}{\sigma_u^2+\sigma_\epsilon^2}$.*

We now analyze how citizens actually behave, given their concern for taking the right action and for social conformity. Because there is a continuum of citizens, a citizen's action does not affect the aggregate outcome, either in the current or in future periods. Thus, the only link between periods is information. From equation (1), citizen i chooses the following degree of social distancing:

$$a_{it} = (1 - \alpha)\mathbb{E}_{it}[\theta_t] + \alpha\ \mathbb{E}_{it}[A_t], \tag{2}$$

where $\mathbb{E}_{it}[\cdot]$ is the expectation of i in period t given his information.

Define $\bar{\mathbb{E}}^h$ recursively as follows. $\bar{\mathbb{E}}^0[X] = X$, $\bar{\mathbb{E}}^1[X] = \bar{\mathbb{E}}[\bar{\mathbb{E}}^0[X]] = \int \mathbb{E}_i[X]di$, $\bar{\mathbb{E}}^h[X] = \bar{\mathbb{E}}[\bar{\mathbb{E}}^{h-1}[X]] = \int \mathbb{E}_i[\bar{\mathbb{E}}^{h-1}[X]]di$. That is, $\bar{\mathbb{E}}^1[X]$ is the average expectation of the random variable X in the population; $\bar{\mathbb{E}}^2[X]$ is the average expectation in the population about the average expectation in the population, and so on. Proposition 2 shows that the aggregate social distancing in the population depends on all such higher order expectations in the population. The proof comes from iterating on equation (2). (All proofs are in the appendix.)

[3] Even if θ_{t-1} is not observed in the current period, citizens will infer it in equilibrium if they observe the last period's aggregate behavior A_{t-1}.

[4] For our normative standard to coincide with the utilitarian social welfare, one can add a term $\alpha \int(a_{jt} - A_t)^2 dj$ to citizen payoffs in equation (1), so that citizens want the divergence of their own behavior from the average behavior to be close to the average divergence in the population.

Proposition 2 ***(Higher Order Beliefs in Aggregate Action)*** *The aggregate action in each period depends on all average higher order beliefs in the population about the right action, with lower weights on higher orders:*

$$A_t = \sum_{h=1}^{\infty}(1-\alpha)\ \alpha^{h-1}\ \bar{\mathbb{E}}_t^h[\theta_t] \tag{3}$$

Observe that

$$\mathbb{E}_{it}[\theta_t] = \mathbb{E}_{it}[u_t] + \theta_{t-1} \ \Rightarrow\ \bar{\mathbb{E}}_t^h[\theta_t] = \bar{\mathbb{E}}_t^h[u_t] + \theta_{t-1}. \tag{4}$$

Now, using properties of Normal distribution and Proposition 2 yields:

Proposition 3 ***(Aggregate Actions Exhibit Excess Inertia)*** *Conformity generates inertia. In particular,*

- $A_t = \theta_{t-1} + \phi u_t$, *where* $0 < \phi < \beta < 1$, $\phi = \frac{(1-\alpha)\beta}{1-\alpha\beta}$, *and* $\beta = \frac{\sigma_u^2}{\sigma_u^2+\sigma_\epsilon^2}$.
- ϕ *is decreasing in* α, *with* $\lim_{\alpha\to 0}\phi(\alpha) = \beta$ *and* $\lim_{\alpha\to 1}\phi(\alpha) = 0$.

Because citizens care about coordinating their actions, they put extra weight on their common knowledge of the right action in the past, which facilitates coordination—reminiscent of the logic of focal points. Citizens have common knowledge that, on average, the right action today is the right action yesterday (i.e., $\theta_t \sim N(\theta_{t-1}, \sigma_u)$) and hence over-weight this fact. As a result, today's aggregate action is biased in the direction of yesterday's right action. As a consequence, following a large positive shock, like COVID-19, to the right amount of social distancing, the aggregate action will be lower than the right action.[5]

Now, suppose in each period t, in addition to the their private signals x_{it}, citizens also receive a public signal $p_t = \theta_t + \eta_t$, with $\eta_t \sim iidN(0, \sigma_\eta)$. Such a public signal might be the result, for instance, of information conveyed by the government. Proposition 2 and equation (4) still hold because they do not depend on the details of available information. However, the presence of public signals changes the degree of inertia in aggregate social distancing.

Proposition 4 ***(Public Announcements)***

1. *Public signals reduce inertia. Averaging over the public signal noise, the expected aggregate social distancing is:* $E[A_t|\theta_{t-1}, u_t] = \theta_{t-1} + \phi_p u_t$, *where* $\phi_p > \phi$.

[5] The over-weighting of public information is a key insight of Morris and Shin (2002) who showed that public announcements can be damaging to welfare, particularly, in financial settings.

2. *The amount of inertia is decreasing in the clarity of the public signal. That is, ϕ_p is monotone decreasing in σ_η, $\lim_{\sigma_\eta \to \infty} \phi_p(\sigma_\eta) = \phi$, and $\lim_{\sigma_\eta \to 0} \phi_p(\sigma_\eta) = 1$.*

3. *Improving the clarity of the public signal causes a normative improvement (i.e., $E[\int (a_{it} - \theta_t)^2 di]$ is increasing in σ_η) if: (i) $\alpha \leq 1/2$ or (ii) σ_η is sufficiently small.*

Proposition 4 shows that, following a shock, an informed leader can send a public signal that helps set public expectations about the aggregate right action, thereby reducing inertia in social distancing driven by the desire to conform. The clearer that message (i.e., the lower σ_η), the more this will reduce inertia.

Such public messages are a normative improvement if people don't put too much weight on conformity ($\alpha \leq 1/2$) or the public signal is sufficiently informative (σ_η small). Why these conditions? Because citizens value conformity, they put excessive weight on all public signals relative to a Bayesian individual who only cares about choosing an action that reflects the best estimate of θ_t (this was the same logic that drove inertia in the first place). Because this distortion is smaller when α is smaller, new public information about the optimal social distancing is always beneficial when citizens put relatively less weight on conformity ($\alpha \leq 1/2$). In the other extreme, when citizens almost only care about conformity ($\alpha \approx 1$), they put almost no weight on their private signals. Now, although citizens over-weight new public information (p_t), this over-reaction to the new public information helps counter-act their over-reaction to past experience (that $\theta_t \sim (\theta_{t-1}, \sigma_u^2)$), and the overall effect is again beneficial. In between, when $\alpha \in (1/2, 1)$, these effects compete and the overall effect of raising the precision of new public information may be negative unless it is sufficiently informative (σ_η small) to offset the over-reaction.[6]

For social distancing in the presence of a dangerous infectious disease, we believe the relevant parameter space is $\alpha \leq 1/2$. It is unlikely that people care so much about conformity that over-reaction to new public information trumps its value. Hence, for cases like COVID-19, Proposition 4 suggests that clear and consistent public messages from a leader are socially beneficial.

Given the overreaction by citizens to public messages described above, one may wonder whether there is a better way for the government to deliver information. Would it be better for citizens to receive the same level of information, but privately rather than publicly? For instance, perhaps employers or local governments could provide private information to citizens, rather than them all observing the same public information in a presidential speech or press conference.

To consider this possibility, contrast the public signal case with a setting where, instead of receiving private and public signals $x_{it} \sim N(\theta_t, \sigma_\epsilon^2)$ and $p_t \sim N(\theta_t, \sigma_\eta^2)$, citizens receive a single private signal x'_{it} with the same amount of information about the right action θ_t

[6] Equation (16) in the proof of Proposition 4 shows the necessary and sufficient conditions for when reducing σ_η is a normative improvement.

as the public and private signals combined. In particular, let $x'_{it} = \theta_t + \epsilon'_{it}$, with $\epsilon'_{it} \sim N(0, \sigma^2_{\epsilon'} = \frac{\sigma^2_\epsilon \sigma^2_\eta}{\sigma^2_\epsilon + \sigma^2_\eta})$.

Proposition 5 *The setting with the combination of private and public signals (x_{it}, p_t) is a normative improvement over the setting with more precise private signals x'_{it} when σ_u is sufficiently small or σ_ϵ is sufficiently large.*

When citizens believe the past is highly informative about the present (σ_u small) or that they are privately poorly-informed (σ_ϵ large), citizens put too much weight on their past experience. In such circumstances, it is better for the government to communicate publicly rather than privately. Citizens over-react to the government's public messages. But that will help to counter-act their over-reaction to their past experience. By contrast, when citizens believe the past is relatively uninformative (σ_u large) or that they are privately well-informed (σ_ϵ small), the government should communicate privately.

In the context of social distancing in the wake of a new and rare infectious disease, individuals' information is typically very noisy (σ_ϵ is large) and the appropriate level of social distancing is very sticky (σ_u is low, the disease is a very unusual shock). As such, for social distancing in the wake of COVID-19, clear and consistent public statements by a national leader are more effective than statements by local governments or employers, not because the national government is more informed, but because clear national statements generate over-reaction that is beneficial to correct the inertia created by the over-weighting of past habits and social norms.

3 References

Angeletos, George-Marios, and Chen Lian. 2016. "Incomplete Information in Macroeconomics: Accommodating Frictions in Coordination." *Handbook of Macroeconomics, Vol. 2*: 1065-1240.

Morris, Stephen and Shin, Hyun Song. 2002. "The Social Value of Public Information." *American Economic Review* 92: 1521-34.

4 Appendix: Proofs

Proof of Proposition 2: From equation (2),

$$A_t = \int a_{it} di = \int ((1-\alpha)\mathbb{E}_{it}[\theta_t] + \alpha\ \mathbb{E}_{it}[A_t]) di = (1-\alpha)\bar{\mathbb{E}}_t[\theta_t] + \alpha\ \bar{\mathbb{E}}_t[A_t].$$

Iterating yields:

$$A_t = (1-\alpha)\bar{\mathbb{E}}_t[\theta_t] + (1-\alpha)\alpha\ \bar{\mathbb{E}}_t^2[\theta_t] + \alpha^2\bar{\mathbb{E}}_t^2[A_t].$$

Repeated iteration yields the result. □

Proof of Proposition 3: We calculate $\bar{\mathbb{E}}_t^h[\theta_t]$, and use Proposition 2. Note that $x_{it} = \theta_t + \epsilon_{it} = \theta_{t-1} + u_t + \epsilon_{it}$. Thus, letting $\beta = \frac{\sigma_u^2}{\sigma_u^2+\sigma_\epsilon^2}$

$$\mathbb{E}_{it}[u_t] = \beta(x_{it} - \theta_{t-1}) = \beta(u_t + \epsilon_{it})\ \Rightarrow\ \bar{\mathbb{E}}_t[u_t] = \beta u_t.$$

Iterating yields:

$$\bar{\mathbb{E}}_t^h[u_t] = \beta^h\ u_t. \tag{5}$$

Substituting from equation (5) into equation (4) yields:

$$\bar{\mathbb{E}}_t^h[\theta_t] = \beta^h\ u_t + \theta_{t-1}. \tag{6}$$

Now, substituting from equation (6) into equation (3) in Proposition 2 yields:

$$A_t = \sum_{h=1}^{\infty}(1-\alpha)\ \alpha^{h-1}\ (\beta^h\ u_t + \theta_{t-1}) = \theta_{t-1} + \frac{\beta(1-\alpha)}{1-\alpha\beta}\ u_t. \tag{7}$$

In equation (7), let $\phi = \frac{(1-\alpha)\beta}{1-\alpha\beta}$, and observe that $\lim_{\sigma_\epsilon\to 0}\beta = \lim_{\sigma_u\to\infty}\beta = 1$. □

Proof of Proposition 4: *Part 1.* With the public signal p_t, $\mathbb{E}_{it}[u_t] = E[u_t|x_{it}, p_t]$. Thus,

$$\begin{aligned}\mathbb{E}_{it}[u_t] &= \frac{\sigma_u^2\sigma_\eta^2(x_{it}-\theta_{t-1}) + \sigma_u^2\sigma_\epsilon^2(p_t-\theta_{t-1})}{\sigma_u^2\sigma_\eta^2 + \sigma_u^2\sigma_\epsilon^2 + \sigma_\epsilon^2\sigma_\eta^2} \\ &= \frac{\sigma_u^2\sigma_\eta^2(u_t+\epsilon_{it}) + \sigma_u^2\sigma_\epsilon^2(p_t-\theta_{t-1})}{\sigma_u^2\sigma_\eta^2 + \sigma_u^2\sigma_\epsilon^2 + \sigma_\epsilon^2\sigma_\eta^2}.\end{aligned} \tag{8}$$

Thus,

$$\bar{\mathbb{E}}_t[u_t] = \frac{\sigma_u^2\sigma_\eta^2 u_t + \sigma_u^2\sigma_\epsilon^2(p_t-\theta_{t-1})}{\sigma_u^2\sigma_\eta^2 + \sigma_u^2\sigma_\epsilon^2 + \sigma_\epsilon^2\sigma_\eta^2} = A_u\ u_t + A_p\ (p_t - \theta_{t-1}), \tag{9}$$

where

$$A_u = \frac{\sigma_u^2\sigma_\eta^2}{\sigma_u^2\sigma_\eta^2 + \sigma_u^2\sigma_\epsilon^2 + \sigma_\epsilon^2\sigma_\eta^2}\ \text{ and }\ A_p = \frac{\sigma_u^2\sigma_\epsilon^2}{\sigma_u^2\sigma_\eta^2 + \sigma_u^2\sigma_\epsilon^2 + \sigma_\epsilon^2\sigma_\eta^2}, \tag{10}$$

with

$$\lim_{\sigma_\eta\to\infty} A_p = 0,\ \lim_{\sigma_\eta\to\infty} A_u = \beta,\ \lim_{\sigma_\eta\to 0} A_p = 1,\ \text{and}\ \lim_{\sigma_\eta\to 0} A_u = 0. \tag{11}$$

Iterating on equation (9) yields

$$\bar{\mathbb{E}}_t^h[u_t] = (A_u)^h\ u_t + (1 + \cdots + A_u^{h-1})A_p(p_t - \theta_{t-1}). \tag{12}$$

Substituting from equation (12) into equation (4) yields:

$$\begin{aligned}\bar{\mathbb{E}}_t^h[\theta_t] &= (A_u)^h\ u_t + (1 + \cdots + A_u^{h-1})A_p(p_t - \theta_{t-1}) + \theta_{t-1} \\ &= (A_u)^h\ u_t + \frac{1 - A_u^h}{1 - A_u} A_p(p_t - \theta_{t-1}) + \theta_{t-1}. \qquad (13)\end{aligned}$$

Now, substituting from equation (13) into equation (3) in Proposition 2 yields:

$$\begin{aligned}A_t &= \sum_{h=1}^{\infty}(1-\alpha)\ \alpha^{h-1}\ \left((A_u)^h\ u_t + \frac{1 - A_u^h}{1 - A_u} A_p(p_t - \theta_{t-1}) + \theta_{t-1}\right) \\ &= \theta_{t-1} + \frac{(1-\alpha)A_u}{1-\alpha A_u}\ u_t + (1-\alpha)\frac{A_p}{1-A_u}\left(\frac{1}{1-\alpha} - \frac{A_u}{1-\alpha A_u}\right)(p_t - \theta_{t-1}) \\ &= \theta_{t-1} + \frac{(1-\alpha)A_u}{1-\alpha A_u}\ u_t + \frac{A_p}{1-\alpha A_u}(p_t - \theta_{t-1}) \\ &= \theta_{t-1} + \frac{(1-\alpha)A_u u_t + A_p(p_t - \theta_{t-1})}{1-\alpha A_u}. \qquad (14)\end{aligned}$$

Note that, using (11), if $\sigma_\eta \to \infty$, equation (14) simplifies to equation (7).

For given θ_{t-1} and u_t, aggregate action A_t takes different values for different values of the public signal p_t, depending on the idiosyncratic error term η_t in the public signal. The average public signal, for given θ_{t-1} and u_t, is $E[p_t|u_t] = \theta_{t-1} + u_t$. Then, averaging over the public signal noise, equation (14) becomes:

$$E[A_t|u_t, \theta_{t-1}] = \theta_{t-1} + \phi_p u_t, \text{ where } \phi_p = \frac{(1-\alpha)A_u + A_p}{1-\alpha A_u}.$$

Part 2. From (11), $\lim_{\sigma_\eta \to 0} \phi_p = 1$ and $\lim_{\sigma_\eta \to \infty} \phi_p = \phi$. Comparing with ϕ in Proposition 3 yields:

$$\phi_p - \phi = \frac{(1-\alpha)A_u + A_p}{1-\alpha A_u} - \frac{(1-\alpha)\beta}{1-\alpha\beta} = \frac{\sigma_\epsilon^4 \sigma_u^2}{(\sigma_\epsilon^2 + (1-\alpha)\sigma_u^2)(\sigma_\epsilon^2\sigma_u^2 + \sigma_\eta^2(\sigma_\epsilon^2 + (1-\alpha)\sigma_u^2))} > 0.$$

That is, $\phi < \phi_p$. Moreover,

$$\frac{d\phi_p}{d\sigma_\eta^2} = -\frac{\sigma_\epsilon^4\sigma_u^2}{\sigma_\epsilon^2\sigma_u^2 + \sigma_\eta^2(\sigma_\epsilon^2 + (1-\alpha)\sigma_u^2))^2} < 0.$$

Thus, reducing the noise in the public signal (less σ_η^2) raises ϕ_p.

Part 3. From (2),

$$\begin{aligned}
a_{it} &= (1-\alpha)\mathbb{E}_{it}[\theta_t] + \alpha\ \mathbb{E}_{it}[A_t] \\
&= (1-\alpha)\mathbb{E}_{it}[\theta_{t-1}+u_t] + \alpha\ \mathbb{E}_{it}\left[\theta_{t-1} + \frac{(1-\alpha)A_u u_t + A_p(p_t-\theta_{t-1})}{1-\alpha A_u}\right] \quad \text{(from (14))} \\
&= \theta_{t-1} + (1-\alpha)\mathbb{E}_{it}[u_t] + \alpha\ \frac{(1-\alpha)A_u}{1-\alpha A_u}\mathbb{E}_{it}\left[u_t\right] + \alpha\frac{A_p(p_t-\theta_{t-1})}{1-\alpha A_u} \\
&= \theta_{t-1} + \frac{1-\alpha}{1-\alpha A_u}\mathbb{E}_{it}\left[u_t\right] + \alpha\frac{A_p(p_t-\theta_{t-1})}{1-\alpha A_u} \\
&= \theta_{t-1} + \frac{1-\alpha}{1-\alpha A_u}(A_u(u_t+\epsilon_{it}) + A_p(p_t-\theta_{t-1})) + \alpha\frac{A_p}{1-\alpha A_u}(p_t-\theta_{t-1}) \quad \text{(from (8) and (10))} \\
&= \theta_{t-1} + \frac{(1-\alpha)A_u}{1-\alpha A_u}u_t + \frac{(1-\alpha)A_u}{1-\alpha A_u}\epsilon_{it} + \frac{A_p}{1-\alpha A_u}(p_t-\theta_{t-1}).
\end{aligned}$$

Thus,

$$\begin{aligned}
a_{it}-\theta_t &= \frac{(1-\alpha)A_u}{1-\alpha A_u}u_t + \frac{(1-\alpha)A_u}{1-\alpha A_u}\epsilon_{it} + \frac{A_p}{1-\alpha A_u}(p_t-\theta_{t-1}) - u_t \\
&= \frac{(A_p+A_u-1)u_t + A_p\ \eta_t + (1-\alpha)A_u\ \epsilon_{it}}{1-\alpha A_u} \quad \text{(substituting } p_t-\theta_{t-1}=u_t+\eta_t\text{)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
(a_{it}-\theta_t)^2 &= \frac{(A_p+A_u-1)^2u_t^2 + A_p^2\ \eta_t^2 + (1-\alpha)^2A_u^2\ \epsilon_{it}^2}{(1-\alpha A_u)^2} \\
&+ \frac{2(A_p+A_u-1)u_tA_p\ \eta_t + 2(A_p+A_u-1)u_t(1-\alpha)A_u\ \epsilon_{it} + 2A_p\ \eta_t(1-\alpha)A_u\ \epsilon_{it}}{(1-\alpha A_u)^2}.
\end{aligned}$$

Thus,

$$\int(a_{it}-\theta_t)^2di = \frac{(A_p+A_u-1)^2u_t^2 + A_p^2\ \eta_t^2 + (1-\alpha)^2A_u^2\ \sigma_\epsilon^2 + 2(A_p+A_u-1)A_p\ u_t\eta_t}{(1-\alpha A_u)^2}.$$

Thus,

$$E[\int(a_{it}-\theta_t)^2di] = \frac{(A_p+A_u-1)^2\sigma_u^2 + A_p^2\ \sigma_\eta^2 + (1-\alpha)^2A_u^2\ \sigma_\epsilon^2}{(1-\alpha A_u)^2}, \tag{15}$$

where we recognize that if $\alpha=0$, equation (15) simplified to $\frac{\sigma_u^2\sigma_\eta^2\sigma_\epsilon^2}{\sigma_u^2\sigma_\eta^2+\sigma_u^2\sigma_\epsilon^2+\sigma_\epsilon^2\sigma_\eta^2}$, which is the variance of $\theta_t|\theta_{t-1},p_t,x_{it}$. Differentiating with respect to σ_η^2 yields:

$$\frac{dE[\int(a_{it}-\theta_t)^2di]}{d\sigma_\eta^2} = \frac{\sigma_\epsilon^4\sigma_u^4(\sigma_\epsilon^2\sigma_u^2+\sigma_\epsilon^2\sigma_\eta^2+(1-\alpha)(1-2\alpha)\sigma_u^2\sigma_\eta^2)}{(\sigma_\epsilon^2\sigma_u^2+\sigma_\eta^2\sigma_\epsilon^2+(1-\alpha)\sigma_\eta^2\sigma_u^2)^3}. \tag{16}$$

Thus, if $\sigma_\epsilon^2 + (1-\alpha)(1-2\alpha)\sigma_u^2 \geq 0$ (in particular, if $\alpha \leq 1/2$), the above derivative is strictly positive. If, instead, $\sigma_\epsilon^2 + (1-\alpha)(1-2\alpha)\sigma_u^2 < 0$, the above derivative is strictly positive if and only if σ_η^2 is sufficiently small. □

Proof of Proposition 5: To obtain $E[\int(a_{it}-\theta_t)^2 di]$ with only x'_{it}, first let $\sigma_\eta \to \infty$ in (15), and then substitute σ_ϵ^2 with $\sigma_{\epsilon'}^2 = \frac{\sigma_\epsilon^2\sigma_\eta^2}{\sigma_\epsilon^2+\sigma_\eta^2}$. Using (11), and recognizing that $\lim_{\sigma_\eta\to\infty} A_p\sigma_\eta^2 = 0$, the first step yields:

$$\lim_{\sigma_\eta\to\infty} \frac{(\beta-1)^2\sigma_u^2 + (1-\alpha)^2\beta^2\ \sigma_\epsilon^2}{(1-\alpha\beta)^2} = \frac{\sigma_\epsilon^2\sigma_u^2(\sigma_\epsilon^2 + (1-\alpha)^2\sigma_u^2)}{(\sigma_\epsilon^2 + (1-\alpha)\sigma_u^2)^2}.$$

Substituting σ_ϵ^2 with $\sigma_{\epsilon'}^2$ yields:

$$\frac{\sigma_{\epsilon'}^2\sigma_u^2(\sigma_{\epsilon'}^2 + (1-\alpha)^2\sigma_u^2)}{(\sigma_{\epsilon'}^2 + (1-\alpha)\sigma_u^2)^2}. \tag{17}$$

Now, subtracting (15) from (17) yields:

$$\Delta = E[\int(a_{it}-\theta_t)^2 di]_{(x'_{it})} - E[\int(a_{it}-\theta_t)^2 di]_{(x_{it},p_t)}$$

$$= \frac{\alpha^2\sigma_\eta^4\sigma_\epsilon^4\sigma_u^4}{(\sigma_\eta^2\sigma_\epsilon^2 + (1-\alpha)(\sigma_\eta^2+\sigma_\epsilon^2)\sigma_u^2)^2(\sigma_\epsilon^2\sigma_u^2 + \sigma_\eta^2(\sigma_\epsilon^2 + (1-\alpha)\sigma_u^2))^2}\ (\sigma_\epsilon^4(\sigma_\eta^2+\sigma_u^2) - (1-\alpha)^2(\sigma_\eta^2+\sigma_\epsilon^2)\sigma_u^4).$$

As expected, $\lim_{\alpha\to 0}\Delta = 0$, because only the amount information matter; and $\lim_{\alpha\to 1}\Delta > 0$, because then citizens put a lot of weight of the pre-existing public information from the previous period, which need to be countered by new public information about θ_t. Moreover, for any $\alpha > 0$, the setting with both public and private signals (x_{it}, p_t) is a normative improvement over the setting with only private signals (x'_{it}) if and only if $\Delta > 0$, i.e., if and only if

$$\left(\frac{\sigma_\epsilon^2}{\sigma_u^2}\right)^2 > (1-\alpha)^2\ \frac{\sigma_\eta^2+\sigma_\epsilon^2}{\sigma_\eta^2+\sigma_u^2}.$$

The result follows from inspection of this inequality. □