Calculus Example Exam Solutions

1. **Limits (18 points, 6 each)**

   Evaluate the following limits:

   (a) $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

   We compute as follows:

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} &= \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} \\
&= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} \\
&= \lim_{x\to 4} \frac{1}{\sqrt{x}+2} \\
&= \frac{1}{\sqrt{4}+2} \\
&= \frac{1}{4}
\end{aligned}$$

   (b) $\lim\limits_{x\to \frac{1}{2}} \dfrac{2x^2+x-1}{2x-1}$

   We compute as follows:

$$\begin{aligned}
\lim_{x\to \frac{1}{2}} \frac{2x^2+x-1}{2x-1} &= \lim_{x\to \frac{1}{2}} \frac{(2x-1)(x+1)}{2x-1} \\
&= \lim_{x\to \frac{1}{2}} x+1 \\
&= \frac{3}{2}
\end{aligned}$$

   (c) $\lim\limits_{x\to +\infty} e^{-x}$

   From basic principles of exponentials, we know that this limit is 0.

2. **Definition of the Derivative (16 points, 10/6)**

   Let $f(x) = x^3 + x$.

   (a) **Use the definition of the derivative to compute $f'(-1)$.**

   We compute from the definition, using the expected algebraic factorization in the fourth line:

$$
\begin{aligned}
f'(-1) &= \lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)} \\
&= \lim_{x\to -1} \frac{(x^3+x)-(-2)}{x+1} \\
&= \lim_{x\to -1} \frac{x^3+x+2}{x+1} \\
&= \lim_{x\to -1} \frac{(x+1)(x^2-x+2)}{x+1} \\
&= \lim_{x\to -1} x^2-x+2 \\
&= 4
\end{aligned}
$$

(b) Write the equation of the line that is tangent to $y = f(x)$ at $x = -1$.

When $x = -1$, we see that $f(-1) = (-1)^3 + (-1) = -2$, so the corresponding point on the curve is $(-1, -2)$. The slope of the line tangent to the curve at that point is given by the derivative, namely $f'(-1) = 4$. Thus, we use Point-Slope Form of a line to find the equation of the tangent line as $y - (-2) = 4(x - (-1))$, which can be simplified to become $y = 4x + 2$.

3. Differentiation (24 points, 8 each)

Differentiate the following functions. You may use any theorems.

(a) $h(x) = \dfrac{1}{\sqrt{1-4x}}$

We re-write the function as $h(x) = (1-4x)^{-1/2}$ and then use the Power Rule and the Chain Rule to get:

$$
\begin{aligned}
h'(x) &= -\tfrac{1}{2}(1-4x)^{-3/2} \cdot (-4) \\
&= \frac{2}{(1-4x)^{3/2}}
\end{aligned}
$$

(b) $j(x) = (1-x^2) \cdot e^{-x^2}$

We use the Product Rule, the Power Rule, and the Chain Rule to get:

$$
\begin{aligned}
j'(x) &= (1-x^2) \cdot e^{-x^2} \cdot (-2x) + (-2x) \cdot e^{-x^2} \\
&= [(1-x^2)+1] \cdot (-2x) \cdot e^{-x^2} \\
&= (2-x^2)(-2x)\, e^{-x^2}
\end{aligned}
$$

(c) $k(x) = (5x^2 + \ln x^4)^{4/3}$

We use the Power Rule, Chain Rule, and Rules for Logarithms to get:

$$
\begin{aligned}
k'(x) &= \frac{4}{3}(5x^2+\ln x^4)^{1/3} \cdot \left(10x + \frac{4x^3}{x^4}\right) \\
&= \frac{4}{3}(5x^2+\ln x^4)^{1/3} \cdot \left(10x + \frac{4}{x}\right)
\end{aligned}
$$

4. **Optimization I (20 points)**

Find all global and local maxima and minima of the function $f(x) = 10 - |x^2 + 2x - 24|$ **on the interval** $[-10, 10]$**.**

The function f is continuous, and the domain $[-10, 10]$ is a closed interval, so a theorem tells us that f will have a global maximum and a global minimum. It is also possible for f to have some local max/mim. Since the absolute value of anything is greater than or equal to zero, we see by inspection that f cannot take on a value greater than 10, but it is not clear that it achieves this maximum value *on this interval*.

We factor the expression inside the absolute value to get: $f(x) = 10 - |(x + 6)(x - 4)|$.

This is very helpful, because it means that when $-6 \leq x \leq 4$, the expression inside the absolute value is negative, and otherwise, the expression is positive. Hence we may re-write the function piecewise:

$$f(x) = \begin{cases} -x^2 - 2x + 34 & , \quad \text{if } -10 \leq x < -6 \\ x^2 + 2x - 14 & , \quad \text{if } -6 \leq x \leq 4 \\ -x^2 - 2x + 34 & , \quad \text{if } 4 < x \leq 10 \end{cases}$$

This enables us to use our usual rules of differentiation to find f' and optimize f.

$$f'(x) = \begin{cases} -2x - 2 & , \quad \text{if } -10 < x < -6 \\ 2x + 2 & , \quad \text{if } -6 < x < 4 \\ -2x - 2 & , \quad \text{if } 4 < x < 10 \end{cases}$$

Note that f is not differentiable at $x = -6$ or at $x = 4$. Nor is it differentiable at the endpoints of the interval, namely $x = -10$ and $x = 10$.

Maxima and minima occur at only at *critical points* which come in three varieties: I. Stationary Points where $f'(x) = 0$, II. Singular Points where f' does not exist, and III. Endpoints. For our function, we have five such points: I. $x = -1$, II. $x = -6$ and $x = 4$, and III. $x = -10$ and $x = 10$.

To find the global maxima and minima, we simply compare the values of the function at the critical points, and our theorem mentioned above guarantees that the max/min among these will be the global max/min. We compute: $f(-10) = -46$, $f(-6) = 10$, $f(-1) = -15$, $f(4) = 10$, and $f(10) = -86$. Clearly, the smallest of these values is -86, so our global minimum occurs at $x = 10$. The largest of these values is 10, so our global maximum occurs twice, at $x = -6$ and again at $x - 4$.

All global max/min are also local max/min, but there may be other local max/min. In fact, at $x = -10$, we have a local min, by the First Derivative Test, and at $x = -1$, we have another local min, by either the First or Second Derivative Test.

5. **Logarithms and Exponentials (20 points, 5/10/5)**

Let $L(a) = k \cdot e^{-\frac{1}{2}(c_1 - a)^2} \cdot e^{-\frac{1}{2}(c_2 - a)^2}$ **for positive constants** k**,** c_1**, and** c_2**.**

(a) Let $l(a) = \ln(L(a))$**. Use the laws of logarithms to write** $l(a)$ **without any exponential functions.**

$$\begin{aligned} l(a) &= \ln(L(a)) \\ &= \ln[k \cdot e^{-\frac{1}{2}(c_1 - a)^2} \cdot e^{-\frac{1}{2}(c_2 - a)^2}] \\ &= \ln k + \ln e^{-\frac{1}{2}(c_1 - a)^2} + \ln e^{-\frac{1}{2}(c_2 - a)^2} \\ &= \ln k - \tfrac{1}{2}(c_1 - a)^2 - \tfrac{1}{2}(c_2 - a)^2 \end{aligned}$$

(b) Compute $\dfrac{dl}{da}$.

Using the expression from part (a) and applying the Power and Chain Rules, we find:

$$\frac{dl}{da} = (c_1 - a) + (c_2 - a)$$

(c) Find all values of a at which $\dfrac{dl}{da} = 0$.

Setting $\frac{dl}{da} = 0$ in part (b) and solving for a, we find $a = \frac{c_1+c_2}{2}$.

6. Analysis of Functions (30 points, 3 each except 6 for part (i))

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the formula:

$$f(x) = \frac{x}{x^2+1}.$$

(a) Compute $f'(x)$.

By the Quotient Rule, we get:

$$f'(x) = \frac{(x^2+1)\cdot 1 - x\cdot(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

(b) Identify all critical points of f.

Since f is defined on all of $\mathbb{R}$, we have no endpoints. Since f' is defined on all of $\mathbb{R}$, we have no singular points. Stationary points occur where $f' = 0$, and the only way a fraction can be zero is when its numerator is, so we find the only critical points to be $x = \pm 1$.

(c) Identify all intervals on which f is increasing and decreasing.

By theorems about derivatives, if $f' > 0$ on an interval, then f is increasing on that interval, and if $f' < 0$, then f is decreasing. Since the denominator of f' is always positive, it suffices to check the sign of the numerator. On the intervals $(-\infty, -1)$ and $(+1, +\infty)$, we see that $f' < 0$, and so f is decreasing. On the interval $(-1, +1)$, we see that $f' > 0$, and so f is increasing.

(d) Identify all local maxima and minima of f.

Local maxima and minima may only occur at critical points. By the First Derivative Test, we see that since f is decreasing on $(-\infty, -1)$ and increasing on $(-1, +1)$, the point $x = -1$ corresponds to a local minimum. Again by the First Derivative Test, we see that since f is increasing on $(-1, +1)$ and decreasing on $(+1, +\infty)$, the point $x = +1$ corresponds to a local maximum.

(e) Compute $f''(x)$.

By the Quotient Rule and Chain Rule, we get:

$$f''(x) = \frac{(x^2+1)^2\cdot(-2x) - (1-x^2)\cdot 2(x^2+1)(2x)}{(x^2+1)^4} = \frac{2x^3-6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

(f) Identify all possible inflection points of f.

The possible inflection points of f occur where f'' is either 0 or undefined. In our case, f'' is always defined, and it is zero only when $x = 0$ or $x = \pm\sqrt{3}$.

(g) Identify all intervals on which f is concave up and concave down.

On the interval $(-\infty, -\sqrt{3})$, we see that $f'' < 0$, so f is concave down on that interval.
On the interval $(-\sqrt{3}, 0)$, we see that $f'' > 0$, so f is concave up on that interval.
On the interval $(0, +\sqrt{3})$, we see that $f'' < 0$, so f is concave down on that interval.
On the interval $(+\sqrt{3}, +\infty)$, we see that $f'' > 0$, so f is concave up on that interval.

(h) **Identify any inflection points of f.**

Since the concavity of f changes on each pair of consecutive intervals, all of the possible inflection points ($x = -\sqrt{3}$, $x = 0$, and $x = +\sqrt{3}$) are actually inflection points.

(i) **Make an accurate graph of $y = f(x)$ on an appropriately scaled set of axes. Make sure the graph illustrates all of the indicated behavior.**

Sorry, cannot easily include graphics in this document. Try Mathematica or a graphing calculator.

7. Partial Derivatives (20 points, 4/6/6/4)

Consider the function $f : S \to \mathbb{R}$ given by the formula:

$$f(x, y) = -xy + 2\ln x + y^2.$$

(a) **Identify the natural domain of f as a subset $S \subset \mathbb{R}^2$.**

Since the natural logarithm is only defined on positive real numbers, the natural domain of the function f is $S = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$.

(b) **Compute $\dfrac{\partial f}{\partial x}$.**

$$\frac{\partial f}{\partial x} = -y + \frac{2}{x}$$

(c) **Compute $\dfrac{\partial f}{\partial y}$.**

$$\frac{\partial f}{\partial y} = -x + 2y$$

(d) **Find all points $(x, y) \in S$ at which $\nabla f(x, y) = (0, 0)$.**

The gradient of the function is the vector of partial derivatives. If $\nabla f(x, y) = (0, 0)$, then we interpret this to mean that $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$. If $\dfrac{\partial f}{\partial y} = 0$, then $-x + 2y = 0$, so $x = 2y$ (*). But if $\dfrac{\partial f}{\partial x} = 0$ as well, then $-y + \dfrac{2}{x} = 0$, so $y = \frac{2}{x}$. Substituting our expression from equation (*), we get $y = \frac{2}{2y}$, which simplifies to $y^2 = 1$, which leads to $y = \pm 1$. Substituting these two values back into equation (*) yields $x = \pm 2$, respectively. Hence there are two points in $\mathbb{R}^2$ where the algebraic expressions for both partial derivatives are zero, namely $(2, 1)$ and $(-2, -1)$. But only the former is in the natural domain S.

8. Optimization II (20 points)

Let $f(x, y) = \alpha x^2 + \beta xy$ for some constants $\alpha, \beta > 0$. Consider the following three constraints:

(1) $x \geq 0$, **(2)** $y \geq 0$, **and (3)** $x + 4y = 5$.

Optimize the function f subject to the constraints.

As this is a function of several variables that we wish to optimize subject to constraints, we use the technique of Lagrange multipliers. The constraint function is $g(x, y) = x + 4y$, and the constraint itself is then a level curve of this function, namely $g = 5$, as given in (3) above. The gradient of the function f is $\nabla f = (2\alpha x + \beta y, \beta x)$. The gradient of the constraint function g is $\nabla g = (1, 4)$. By the Lagrange multipliers theorem, the function f will be optimized subject to the constraint $g = 5$ when there is some constant (the Lagrange multiplier) $\lambda \in \mathbb{R}$ such that $\nabla f = \lambda \nabla g$. This leads to the vector equation $(2\alpha x + \beta y, \beta x) = \lambda(1, 4)$. This vector equation should be read as two separate equations:

$$2\alpha x + \beta y = \lambda$$

$$\beta x = 4\lambda$$

These are two equations in the three variables x, y, and λ. (Note that α and β are pre-determined constants.) To solve this system, we need a third equation, which we have in the form of the constraint $g = 5$, which should be written in its original form as:

$$x + 4y = 5$$

Combining the first two equations by substituting the value of λ from the first into the second yields:

$$\beta x = 4(2\alpha x + \beta y)$$

Or:

$$(\beta - 8\alpha)x = 4\beta y$$

Taking $x = 5 - 4y$ from the constraint equation and substituting it into this last equation yields:

$$(\beta - 8\alpha)(5 - 4y) = 4\beta y$$

This can then be simplified and solved to get $y = \dfrac{5\beta - 40\alpha}{8\beta - 32\alpha}$.

Plugging this back in to get the other variable, we also find $x = \dfrac{20\beta}{8\beta - 32\alpha}$.

Finally, we must decide whether this point $\left(\dfrac{20\beta}{8\beta - 32\alpha}, \dfrac{5\beta - 40\alpha}{8\beta - 32\alpha}\right)$ is a maximum, a minimum, or neither. The only other places where f may attain its max or min are at the endpoints of the constraint set, given by (1), (2), and (3) combined, namely, at the two points $(5, 0)$ and $(0, \frac{5}{4})$. It is clear from the definition of f that $f(0, \frac{5}{4}) = 0$, which must be a minimum since f takes only non-negative values when $x, y \geq 0$. At the other end, $f(5, 0) = 25\alpha$. We compare this to our newly discover point, where

$$f\left(\frac{20\beta}{8\beta - 32\alpha}, \frac{5\beta - 40\alpha}{8\beta - 32\alpha}\right) = \frac{-400\alpha\beta^2 + 100\beta^3}{(8\beta - 32\alpha)^2} = \frac{\beta^2(100\beta - 400\alpha)}{(8\beta - 32\alpha)^2} = \frac{25\beta^2}{16(\beta - 4\alpha)}.$$

Whether this is a maximum or a minimum depends on the values of α and β.

9. Definite Integrals (10 points)

Find the value of the constant β such that $\displaystyle\int_1^2 (x^2 + \beta x)\, dx = 4$.

We compute the definite integral on the left-hand side as follows:

$$\begin{aligned} \textstyle\int_1^2 (x^2 + \beta x)\, dx &= \left[\tfrac{1}{3}x^3 + \tfrac{\beta}{2}x^2\right]_{x=1}^{x=2} \\ &= [\tfrac{8}{3} + 2\beta] - [\tfrac{1}{3} + \tfrac{\beta}{2}] \\ &= \tfrac{7}{3} + \tfrac{3\beta}{2} \end{aligned}$$

If this is equal to 4, we get $4 = \frac{7}{3} + \frac{3\beta}{2}$, which can be solved to get $\beta = \frac{10}{9}$.