Calculus Example Exam

We strongly encourage students to sit this exam as if it were the real thing – take it in a quiet setting all at one time, avoid using notes or calculators, and enforce the time limit. Use the scoring guide to grade your own paper and check whether you would pass the exam.

Instructions:

- This exam is closed book, closed notes, no calculators.
- You have 1½ hours to complete the exam.
- Show your work on a separate paper – there is partial credit given on the real tests!
- Your answers should be simplified

SCORE: _____________ / 178 points

The exam has the following nine sections. Scores of 60% or better will pass the exam. Please budget your time accordingly.

Limits:	18 pts
Definition of the Derivative:	16 pts
Differentiation:	24 pts
Optimization I:	20 pts
Logarithms and Exponentials:	20 pts
Analysis of Functions:	30 pts
Partial Derivatives:	20 pts
Optimization II:	20 pts
Definite Integrals:	10 pts

1. **Limits** (18 points, 6 each)

   Evaluate the following limits:

   (a) $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

   (b) $\lim_{x\to \frac{1}{2}} \dfrac{2x^2+x-1}{2x-1}$

   (c) $\lim_{x\to +\infty} e^{-x}$

2. **Definition of the Derivative** (16 points, 10/6)

   Let $f(x) = x^3 + x$.

   (a) Use the definition of the derivative to compute $f'(-1)$.

   (b) Write the equation of the line that is tangent to $y = f(x)$ at $x = -1$.

3. **Differentiation** (24 points, 8 each)

   Differentiate the following functions. You may use any theorems.

   (a) $h(x) = \dfrac{1}{\sqrt{1-4x}}$

   (b) $j(x) = (1-x^2)\cdot e^{-x^2}$

   (c) $k(x) = (5x^2 + \ln x^4)^{4/3}$

4. **Optimization I** (20 points)

   Find all global and local maxima and minima of the function $f(x) = 10 - |x^2 + 2x - 24|$ on the interval $[-10, 10]$.

5. **Logarithms and Exponentials** (20 points, 5/10/5)

   Let $L(a) = k \cdot e^{-\frac{1}{2}(c_1-a)^2} \cdot e^{-\frac{1}{2}(c_2-a)^2}$ for positive constants k, c_1, and c_2.

   (a) Let $l(a) = \ln(L(a))$. Use the laws of logarithms to write $l(a)$ without any exponential functions.

   (b) Compute $\dfrac{dl}{da}$.

   (c) Find all values of a at which $\dfrac{dl}{da} = 0$.

6. Analysis of Functions (30 points, 3 each except 6 for part (i))

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the formula:

$$f(x) = \frac{x}{x^2 + 1}.$$

(a) Compute $f'(x)$.

(b) Identify all critical points of f.

(c) Identify all intervals on which f is increasing and decreasing.

(d) Identify all local maxima and minima of f.

(e) Compute $f''(x)$.

(f) Identify all possible inflection points of f.

(g) Identify all intervals on which f is concave up and concave down.

(h) Identify any inflection points of f.

(i) Make an accurate graph of $y = f(x)$ on an appropriately scaled set of axes. Make sure the graph illustrates all of the indicated behavior.

7. Partial Derivatives (20 points, 4/6/6/4)

Consider the function $f : S \to \mathbb{R}$ given by the formula:

$$f(x, y) = -xy + 2 \ln x + y^2.$$

(a) Identify the natural domain of f as a subset $S \subset \mathbb{R}^2$.

(b) Compute $\dfrac{\partial f}{\partial x}$.

(c) Compute $\dfrac{\partial f}{\partial y}$.

(d) Find all points $(x, y) \in S$ at which $\nabla f(x, y) = (0, 0)$.

8. Optimization II (20 points)

Let $f(x, y) = \alpha x^2 + \beta xy$ for some constants $\alpha, \beta > 0$. Consider the following three constraints:

(1) $x \geq 0$, (2) $y \geq 0$, and (3) $x + 4y = 5$.

Optimize the function f subject to the constraints.

9. Definite Integrals (10 points)

Find the value of the constant β such that $\displaystyle\int_1^2 (x^2 + \beta x)\, dx = 4$.